

A Strongly Exponential Separation of DNNFs from CNF Formulas

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Abstract

Decomposable Negation Normal Forms (DNNFs) are Boolean circuits in negation normal form where the subcircuits leading into each AND gate are defined on disjoint sets of variables. We prove a strongly exponential lower bound on the size of DNNFs for a class of CNF formulas built from expander graphs. As a corollary, we obtain a strongly exponential separation between DNNFs and CNF formulas in prime implicants form. This settles an open problem in the area of knowledge compilation (Darwiche and Marquis, 2002).

1 Introduction

The aim of knowledge compilation is to succinctly represent propositional knowledge bases in a format that allows for answering a number of queries in polynomial time [DM02]. Choosing a representation language generally involves a trade-off between succinctness and the range of queries that can be efficiently answered. For instance, CNF formulas are more succinct than prime implicate formulas (PIs), but the latter representation enjoys clause entailment checks in polynomial time whereas CNF formulas in general do not, unless $P = NP$ [GKPS95, CM78]. The need to balance the competing requirements of succinctness and tractability has led to the introduction of a large variety of representation languages that strike this balance in different ways.

Decomposable Negation Normal Forms (DNNFs) are Boolean circuits in negation normal form (NNF) such that the subcircuits leading into an AND gate are defined on disjoint sets of variables [Dar01]. DNNFs are among the most succinct representation languages considered in knowledge compilation—for instance, they generalize variants of binary decision diagrams such as ordered binary decision diagrams (OBDDs) and even free binary decision diagrams (FBDDs, also known as read-once branching programs). They have also been studied in circuit complexity, under the name of multilinear Boolean circuits [SV94, PV04, Kri07].

In this paper, we consider the relative succinctness of DNNFs and CNF formulas. On the one hand, DNNFs can be exponentially more succinct than CNF formulas [GKPS95]. On the other hand, Darwiche and Marquis observed that CNFs do not admit polynomial DNNF representations unless the polynomial hierarchy collapses, while posing an unconditional proof of such a separation as an open problem [DM02].

An unconditional, *weakly exponential* separation can be derived from known results (see the section on related work below). By using a more direct construction that leverages the combinatorial properties of expander graphs, we obtain a *strongly exponential* separation (Theorem 5):

There is a class \mathcal{C} of CNF formulas such that for each $F \in \mathcal{C}$, the DNNF size of F is $2^{\Omega(n)}$, where n is the number of variables of F .

The formulas in \mathcal{C} satisfy strong syntactic restrictions. In particular, they are in prime implicate form, so we immediately obtain an exponential separation of DNNFs from PIs (Corollary 3), answering an open question by Darwiche and Marquis [DM02].

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Our result further improves the best known lower bound of $\Omega\left(\frac{2^{\sqrt{n}}}{\sqrt[3]{n}}\right)$ on the DNNF size of a Boolean function of n variables [Kri07].

Related Work. We observe that an unconditional, weakly exponential separation of DNNFs from CNF formulas can be obtained from known results as follows. Let F be a CNF formula encoding the run of a nondeterministic polynomial-time Turing machine deciding the clique problem for some fixed input size. A DNNF representation of F can be turned into a DNNF computing the clique function by projecting on the variables encoding the input. This can be done without increasing the size of the DNNF [Dar01]. Since an optimal DNNF computing a monotone function is monotone [Kri07], weakly exponential lower bounds for monotone circuits computing the clique function [AB87] transfer to lower bounds on the DNNF size of F .

The formulas used to prove our main result are based on expander graphs and were originally introduced to establish an exponential lower bound for the OBDD size of CNFs [BS14]. The present paper leverages a recent result by Razgon [Raz14b, Theorem 4] to lift this lower bound to DNNFs; indeed, we slightly improve Razgon’s result, while at the same time providing a significantly shorter proof based on a new combinatorial result (Theorem 3).

There is a rich literature on lower bounds for more restricted representation languages, such as OBDDs or FBDDs [Weg00, Juk12]. Moreover, certain subclasses of DNNFs, so-called decision-DNNFs, have been recently considered in database theory in the context of probabilistic databases. In this setting, lower bounds are obtained by a quasipolynomial simulation of decision-DNNFs by FBDDs in combination with known exponential lower bounds for FBDDs [BLRS13, BLRS14]. Pipatsrisawat and Darwiche have proposed a framework for showing lower bounds on structured DNNFs [PD10], a subclass of DNNFs in which the variables respect a common tree-ordering.

2 Preliminaries

Let X be a countable set of variables. A literal is a variable (x) or a negated variable ($\neg x$). An assignment is a function f from X to the constants 0 and 1. We occasionally identify an assignment f with the set of literals $\{\neg x: f(x) = 0\} \cup \{x: f(x) = 1\}$.

NNFs. A *negation normal form (NNF)* C (also known as a *De Morgan* circuit) is a node labeled directed acyclic graph (DAG), whose labeled nodes and arcs are respectively called the *gates* and *wires* of C . The underlying DAG has a unique sink (outdegree 0) node, referred to as the *output gate* of the circuit, and denoted by $\text{output}(C)$. The source nodes of C (indegree 0), denoted by $\text{inputs}(C)$, are referred to as the *input gates* of C and are labeled by a constant (0 or 1) or by a literal x or $\neg x$ for $x \in X$. We let $\text{vars}(C)$ denote the set of variables occurring in the labels of input gates of C . The non source nodes of C , referred as *internal gates*, are labeled by \wedge or \vee . In this paper, the *size* of C , in symbols $\text{size}(C)$, is the number of wires in C .

Let G be a DAG and let v be a node in G . The *subgraph of G sinked at v* is the DAG whose node set is

$$V' = \{v\} \cup \{u: \text{there exists a directed path from } u \text{ to } v \text{ in } G\},$$

and whose arcs are exactly the arcs of G among the nodes in V' . Let C be an NNF, and let v be a node in the DAG G underlying C . We let $\text{sub}(C, v)$ denote the *subcircuit of C sinked at v* , that is, the NNF whose underlying DAG is the subgraph of G sinked at v , with the same labels, and whose variables are those labeling the input gates of $\text{sub}(C, v)$.

Let C be an NNF, v be a gate in C , and f be an assignment. The *value of $\text{sub}(C, v)$ under f* , in symbols $\text{sub}(C, v)(f)$, is defined inductively as usual. The *value of C under f* , in symbols $C(f)$, is equal to $\text{sub}(C, \text{output}(C))(f)$. We let $\text{sat}(C)$ denote the set of *satisfying assignments* of C , that is, assignments f such that $C(f) = 1$. Let C be an NNF on variables $\text{vars}(C)$. Two NNFs C and C' are *equivalent* if

$$C(f) = C'(f)$$

for all assignments f .

Certificates. Let C be an NNF. A *certificate* of C is an NNF T whose gates and wires are subsets of the gates and wires of C satisfying the following:

- $\text{output}(C) = \text{output}(T)$;
- if a gate v is in $C \cap T$, and v is a \wedge -gate in C with input wires from gates v_1, \dots, v_i , then the gates v_1, \dots, v_i and the wires $(v_1, v), \dots, (v_i, v)$ are in T ($i \geq 0$);
- if a gate v is in $C \cap T$, and v is an \vee -gate in C with input wires from gates v_1, \dots, v_i , then exactly one gate $w \in \{v_1, \dots, v_i\}$ is in T and the wire (w, v) is in T ($i \geq 0$).

We let $\text{cert}(C)$ denote the set of certificates of C .

Satisfying assignments and certificates of an NNF are nicely related as follows.

Proposition 1. *Let C be an NNF and let f be an assignment. Then, $f \in \text{sat}(C)$ if and only if there exists $T \in \text{cert}(C)$ such that $f \in \text{sat}(T)$.*

Proof. For the forward direction, let $f \in \text{sat}(C)$. Call a wire (u, v) in the DAG underlying C *activated* by f if f satisfies the subcircuit of C sinked at u , in symbols $\text{sub}(C, u)(f) = 1$. It is readily verified that there exists a certificate T for C containing only wires activated by f . Moreover, $f \in \text{sat}(T)$ because by construction $\text{sub}(T, v)(f) = 1$ for all input gates v of T , and therefore $\text{sub}(T, \text{output}(T))(f) = 1$.

For the backward direction, let T be a certificate of C such that $f \in \text{sat}(T)$. By induction on the structure of C , we prove that for all $t \in T$ it holds that $\text{sub}(T, t)(f) = \text{sub}(C, t)(f) = 1$. Since $\text{output}(C) = \text{output}(T)$, we conclude that

$$C(f) = \text{sub}(C, \text{output}(C))(f) = \text{sub}(T, \text{output}(T))(f) = 1.$$

Then, $f \in \text{sat}(C)$.

If t is an input gate of T , then $\text{sub}(T, t)(f) = 1$ because otherwise f does not satisfy T . We also have that t is an input gate of C , hence $\text{sub}(C, t)(f) = 1$. Let t be a \vee -gate in T , with input wires from gates t_1, \dots, t_i in C ; say without loss of generality that t_1 is chosen in T . By the induction hypothesis, $\text{sub}(T, t_1)(f) = \text{sub}(C, t_1)(f) = 1$; hence, $\text{sub}(T, t)(f) = \text{sub}(C, t)(f) = 1$. The case where t is a \wedge -gate is similar. \square

DNNFs and CNFs. An NNF D is *decomposable* (in short, a *DNNF*) if for all \wedge -gates v with input wires from gates v_1, \dots, v_i and all $j, j' \in \{1, \dots, i\}$, $j \neq j'$, the variable sets of the subcircuits of D sinked at v_j and $v_{j'}$ are disjoint, in symbols,

$$\text{vars}(\text{sub}(D, v_j)) \cap \text{vars}(\text{sub}(D, v_{j'})) = \emptyset.$$

A *conjunctive normal form* (in short, *CNF*) is a finite conjunction of clauses (finite disjunctions of literals). Equivalently, a CNF is an NNF where the maximum number of wires on a path from an input gate to the output gate is 2, and each \vee -gate has input wires only from input gates. A CNF F is *monotone* if its labels do not contain negative literals, a *k*-CNF if the fanin of \vee -gates is at most k , and a *read k times* CNF if, for every $x \in \text{vars}(F)$, the number of wires leaving nodes whose label contain the variable x is at most k .

For a CNF E , we denote by $\text{DNNF}(E)$ the size of the smallest DNNF equivalent to E , that is

$$\text{DNNF}(E) = \min\{\text{size}(D) : D \text{ is a DNNF equivalent to } E\}.$$

Graphs. We refer to a standard reference for basic notions and facts in graph theory [Die05]. Let $G = (V, E)$ be a graph. A *vertex cover* of G is a subset C of the vertices V such that $\{u, v\} \cap C \neq \emptyset$ for all $\{u, v\} \in E$. We denote by $\text{VC}(G)$ the set of the vertex covers of G .

We observe two facts about graphs that will be useful later in proving the main result. The first is that vertex covers of graphs in a class of graphs of bounded degree are large.

Proposition 2. *Let $C \in \text{VC}(G)$ be a vertex cover of a connected graph $G = (V, E)$ of degree d . Then, $|V|/(d+1) \leq |C|$.*

Proof. Let C be a vertex cover of a connected graph G of maximum degree d . Then, $V \setminus C$ is an independent set. Since G is connected, each vertex in $V \setminus C$ is incident to at least one edge with a vertex in C . Hence, there are at least $|V \setminus C|$ edges between C and $V \setminus C$. Since each vertex in C has degree at most d ,

$$|V| - |C| = |V \setminus C| \leq d|C|,$$

and we are done. \square

The second is that a rooted binary tree with a large number of leaves always contains a subtree with a large but not too large number of leaves.

Proposition 3. *Let T be a rooted binary tree with at least ℓ leaves. Then there exists a vertex v of T such that the number of leaves of the subtree of T rooted in v is at least ℓ and at most 2ℓ .*

Proof. The proof is by induction on the size of T . If the number of leaves of T is already between ℓ and 2ℓ , then we can choose the root. Now if T has more than 2ℓ leaves then either the root has one child w . In this case, we apply the induction hypothesis on the subtree of T rooted in w , since it has also more than 2ℓ leaves, that is more than ℓ leaves.

Now assume that the root has two children w_1, w_2 . Let T_1 and T_2 be the subtrees rooted in w_1 and w_2 respectively. Assume without loss of generality that T_1 has more leaves than T_2 . Thus T_1 has more than $2\ell/2 = \ell$ leaves. By induction, there exists a vertex v in T_1 and thus in T such that the subtree rooted in v has at least ℓ and at most 2ℓ leaves. \square

Expanders. Let $G = (V, E)$ be a graph. For every $S \subseteq V$, we let $N_{S,G}$ denote the *open neighbourhood* of S in G , in symbols,

$$N_{S,G} = \{v \in V \setminus S : \text{there exists } u \in S \text{ such that } \{u, v\} \in E\};$$

we write N_v instead of $N_{\{v\},G}$ ($v \in V$), and N_S instead of $N_{S,G}$ if the intended graph G is clear from the context. For a vertex $v \in V$, we denote the degree of v by $d(v) = |N_v|$; the degree of G is the maximum degree attained over its vertices.

Let $d \geq 3$ and $c > 0$. A graph $G = (V, E)$ is a (c, d) -*expander* if G has degree d and for all $S \subseteq V$ such that $|S| \leq |V|/2$ it holds that

$$|N_{S,G}| \geq c|S|. \tag{1}$$

Note that a (c, d) -expander is connected and that taking $|S| = |V|/2$ implies that $c \leq 1$.

Theorem 1 (Section 9.2 in [AS00]). *For all $d \geq 3$, there exists $c > 0$ and a sequence of graphs $\{G_i \mid i \in \mathbb{N}\}$ such that $G_i = (V_i, E_i)$ is a (c, d) -expander and $|V_i| \rightarrow \infty$ as $i \rightarrow \infty$ ($i \in \mathbb{N}$).*

3 Outline of the Proof

We consider a class of what we call graph CNFs. A *graph CNF* is a monotone 2-CNF corresponding to a graph in that a clause $x \vee y$ in the CNF corresponds to an edge $\{x, y\}$ in the graph. Note that the models of a graph CNF correspond exactly to the vertex covers of the underlying graph.

More specifically, our graph CNFs correspond to an infinite family of *expander graphs*. A graph $G = (V, E)$ in this family is highly connected but sparse (in the sense of Theorem 1). As a consequence, given any $S \subseteq V$ of size no larger than $|V|/2$, but linear in $|V|$, it is possible to find a matching of size linear in $|V|$ between S and $V \setminus S$ (Corollary 1). This is crucial to establish a strongly exponential lower bound.

An optimal DNNF computing a monotone Boolean function is monotone [Kri07, Lemma 3]. Since graph CNFs are monotone, it suffices to prove a lower bound for monotone DNNFs (Proposition 4). We do this by means of a *bottleneck counting* argument [Hak85]: we identify a set B of gates such that each satisfying assignment of the DNNF has to pass through one of these gates, and argue that the number of assignments

passing through an individual gate is small. Since the number of satisfying assignments is large, we conclude that the number of gates must be large as well.

More specifically, the argument goes as follows. Let F be a graph CNF whose underlying graph $G = (V, E)$ is an expander (of degree d), and let D be a (nice) DNNF computing F . The set B of gates is defined by taking, for every certificate T of D , a gate v_T in T such that the number of variables in the subcircuit of D rooted at v_T is between $|V|/(d+1)$ and $|V|/2$ (Lemma 2). It follows from the expansion properties of G (Lemma 3) and the decomposability properties of D (Theorem 2 and Corollary 1) that for all gates $v \in B$ there exists a subset I_v of V of size linear in $|V|$ such that, for all certificates T of D containing the gate v , it holds that I_v is contained in the variables of T . The satisfying assignments of D passing through v are those mapping all variables in I_v to 1.

Next, we show that for every $v \in B$, the fraction of satisfying assignments of D containing I_v is exponentially small in $|V|$ (Theorem 3 and Corollary 2). Moreover, the union (over gates $v \in B$) of satisfying assignments of D mapping I_v to 1 coincides with the satisfying assignments of D . It follows that the size of B is exponentially large in $|V|$ (Theorem 4).

4 Proof of the Lower Bound

In this section, we prove our main result. We introduce graph CNFs and nice DNNFs, prove a key property of nice DNNFs computing graph CNFs (Section 4.1), and present our bottleneck argument (Section 4.2).

4.1 Graph CNFs and Nice DNNFs

If G is a graph with at least two vertices and no isolated vertices, we view the edge set of G as a CNF on the variables $\text{vars}(E) = V$, namely,

$$\bigwedge_{\{x,y\} \in E} (x \vee y); \quad (2)$$

we call a CNF of the form (2) a *graph CNF*, and identify it with E .

Note that the satisfying assignments of a graph CNF E correspond to vertex covers of the underlying graph $G = (V, E)$ as follows: If f is a satisfying assignment of E , then $\{x \in V : f(x) = 1\} \in \text{VC}(G)$, and if $V' \in \text{VC}(G)$, then any assignment f such that $V' \subseteq \{x \in V : f(x) = 1\}$ satisfies E .

An NNF is called *negation free* if no input gate is labeled by a negated variable ($\neg x$), and *constant free* if no input gate is labeled by a constant (0 or 1). Note that, if C is a negation and constant free NNF, then $\text{vars}(C)$ coincides with the labels of the input gates of C . A fanin 2, constant free, and negation free DNNF is called *nice*.

The following statement implies that the minimum size of a nice DNNF computing a graph CNF (but indeed, more generally, any monotone Boolean function) is at most 2 times as large as its DNNF size.

Proposition 4. *Let E be a graph CNF and let D be a DNNF equivalent to E . There exists a nice DNNF D' equivalent to D such that $\text{size}(D') \leq 2 \cdot \text{size}(D)$.*

We first reduce to the fanin 2 case.

Proposition 5. *Let D be a DNNF. There exists a DNNF D' equivalent to D , having fanin 2, and such that $\text{size}(D') \leq 2 \cdot \text{size}(D)$.*

Proof. Let D be a DNNF. An NNF D' equivalent to D and having fanin 2 is obtained by editing D as follows, until no gate of fanin larger than 2 exists: Let v be a \wedge -gate with input wires from gates v_1, \dots, v_i with $i > 2$; delete the wires (v_j, v) for $j \in \{2, \dots, i\}$; create a fresh \wedge -gate w , and the wires (w, v) and (v_j, w) for $j \in \{2, \dots, i\}$. The case where v is an \vee -gate is similar.

It is readily verified that D' is decomposable. Moreover, each wire in D is processed at most once (when it is an input wire of a gate having fanin larger than 2) and it generates at most 2 wires in D' , hence the size of D' is at most twice the size of D . \square

Next, we reduce to the negation free case. A Boolean function $F: \{0,1\}^Y \rightarrow \{0,1\}$ is called *monotone* if for all assignments $f, f' \in Y \rightarrow \{0,1\}$ such that $f(x) \leq f'(x)$ for all $x \in Y$, it holds that $F(f) \leq F(f')$.

Proposition 6. [Kri07, Lemma 3] *Let D be a DNNF computing a monotone Boolean function. There exists a DNNF D' equivalent to D , negation free, and such that $\text{size}(D') \leq \text{size}(D)$. Moreover, D' has the same fanin as D .*

Proof. Suppose D contains a gate u labeled with literal $\neg x$. Let D' be the DNNF on $\text{vars}(D)$ obtained from D by relabeling u with the constant 1. We claim that an assignment satisfies D if and only if it satisfies D' . Let f be a satisfying assignment of D . By Proposition 1 there is a certificate T of D such that f satisfies T . We obtain a certificate T' of D' by relabeling u with the constant 1 (if u appears in T). It is straightforward to verify that f is a satisfying assignment of T' . We apply Proposition 1 once more to conclude that f must be a satisfying assignment of D' . For the converse, let f be a satisfying assignment of D' , and let T' be a certificate of D' such that f satisfies T' . If T' does not contain the gate u then T' is also a certificate of D , and f is a satisfying assignment of D by Proposition 1. Otherwise, we obtain a certificate T of D from T' by relabeling u with the literal $\neg x$. Let f' be the assignment such that $f(y) = f'(y)$ for all $y \in \text{vars}(D) \setminus \{x\}$, and such that $f'(x) = 0$. Since D is decomposable and T contains the gate u labeled with $\neg x$, no node of T can be labeled with the literal x . Thus T' cannot contain such a gate either, and f' satisfies T' . Since $\neg x$ evaluates to 1 under f' , the certificate T is satisfied by f' as well. By Proposition 1, the assignment f' is a satisfying assignment of D . Because the function computed by D is monotone, we conclude that f must satisfy D as well. Clearly D and D' have the same size and maximum fanin. It follows that the desired negation free DNNF can be obtained by replacing every negative literal by the constant 1 in the labels of D . \square

Finally, we reduce to the constant free case. Let D be a DNNF not equivalent to 0 or 1. A constant free DNNF, denoted by $\text{elimconst}(D)$, is obtained by editing D as follows, until all gates labeled by a constant are deleted: Let v be a 0-gate, and let v have wires to gates v_1, \dots, v_r . For all $j \in [r]$: if v_j is a \wedge -gate, relabel v_j by 0, and delete all the input wires of v_j (possibly creating some undesigned sink nodes in the underlying DAG); if v_j is a \vee -gate, then delete the wire (v, v_j) ; relabel v_j by 0 if it becomes fanin 0; finally, delete v . The case where v is a 1-gate is similar. Clearly,

Proposition 7. *Let D be a non-constant DNNF and let $D' = \text{elimconst}(D)$. Then, D and D' are equivalent, and $\text{size}(D') \leq \text{size}(D)$; moreover, the fanin and negation freeness of D are preserved in D' .*

We conclude proving the statement.

Proof of Proposition 4. Let E be a graph CNF and let D be a DNNF equivalent to E . By Proposition 5, there exists a DNNF D_1 , equivalent to D , having fanin 2 whose size is at most twice the size of D . Since E is a monotone Boolean function, by Proposition 6, there exists a fanin 2 and negation free DNNF D_2 , equivalent to D_1 , whose size is at most the size of D_1 . Since E is a non constant Boolean function, by Proposition 7, there exists a fanin 2, negation free, and constant free DNNF D_3 , equivalent to D_2 , whose size is at most the size of D_3 . Let $D' = D_3$. Then, D' is a fanin 2, constant free, and negation free DNNF, equivalent to D , whose size is at most twice the size of D . \square

We also observe that, because of decomposability, certificates of nice DNNFs are tree shaped.

Proposition 8. *Let D be a (fanin 2) constant free DNNF and let $T \in \text{cert}(D)$. The undirected graph underlying T is a (binary) tree. Moreover, no two leaves of T are labeled by the same variable.*

Proof. Assume that the undirected graph underlying T is cyclic, so that in the underlying DAG there exist two distinct nodes v and w in T and two arc disjoint directed paths from v to w ; in particular, w has at least two ingoing arcs in T , hence by construction w is a \wedge -gate in D . By decomposability, no variables occur as labels of input gates in $\text{sub}(D, v)$, which is impossible since D is constant free.

We now consider that T is rooted in $\text{output}(D)$. Let ℓ_1 and ℓ_2 be two distinct leaves of T . Their least common ancestor w in T has two ingoing arcs, thus it is an \wedge -gate. By decomposability of w in D , ℓ_1 and ℓ_2 are labeled by a different variable. \square

Let D be a nice DNNF computing a graph CNF E with underlying graph $G = (V, E)$, and let $\{x, x'\}$ be an edge (clause) in E such that for some gate v in D , the variables x and x' are, respectively, inside and outside the subcircuit of D rooted at v . In this case, as we now show, all certificates of D through v set x to 1, or all certificates of D through v set x' to 1.

Theorem 2. *Let D be a nice DNNF, $v \in D$, $x \in \text{vars}(\text{sub}(D, v))$, and $x' \in \text{vars}(D) \setminus \text{vars}(\text{sub}(D, v))$. If $\{x, x'\} \cap \text{vars}(T) \neq \emptyset$ for all $T \in \text{cert}(D)$, then at least one of the following two statements holds:*

- $x \in \text{vars}(T)$ for all $T \in \text{cert}(D)$ such that $v \in T$.
- $x' \in \text{vars}(T)$ for all $T \in \text{cert}(D)$ such that $v \in T$.

Proof. Let $\{T, T'\} \subseteq \text{cert}(D)$ be such that $v \in T \cap T'$. As D is constant free, by Proposition 8, the underlying graphs of the certificates of D are trees. By hypothesis, $\{x, x'\} \cap \text{vars}(T) \neq \emptyset$ and $\{x, x'\} \cap \text{vars}(T') \neq \emptyset$. We want to show that $x \in \text{vars}(T) \cap \text{vars}(T')$ or $x' \in \text{vars}(T) \cap \text{vars}(T')$.

Assume towards a contradiction that $x \in \text{vars}(T) \setminus \text{vars}(T')$ and $x' \in \text{vars}(T') \setminus \text{vars}(T)$; the case where $x' \in \text{vars}(T) \setminus \text{vars}(T')$ and $x \in \text{vars}(T') \setminus \text{vars}(T)$ is symmetric.

First, we observe that $x \notin \text{vars}(T) \setminus \text{vars}(\text{sub}(T, v))$ since, by Proposition 8, the leaves of T are labeled with distinct variables and $x \in \text{vars}(\text{sub}(T, v))$.

Second, since $\text{vars}(\text{sub}(T', v)) \subseteq \text{vars}(\text{sub}(D, v))$, and $x' \notin \text{vars}(\text{sub}(D, v))$ by hypothesis, it holds that $x' \notin \text{vars}(\text{sub}(T', v))$. Therefore,

$$\{x, x'\} \cap \text{vars}(T) \setminus \text{vars}(\text{sub}(T, v)) = \emptyset \text{ and } \{x, x'\} \cap \text{vars}(\text{sub}(T', v)) = \emptyset.$$

Now, the tree S obtained by replacing in T the subtree rooted at v by the subtree rooted at v in T' is a certificate of D ; moreover, $\{x, x'\} \cap \text{vars}(S) = \emptyset$, contradicting the hypothesis that all certificates of D have a nonempty intersection with $\{x, x'\}$. \square

Therefore, if G contains a matching M such that each edge in the matching satisfies the condition of the previous statement, namely there is a gate v in D such that each edge in M has one vertex inside and the other vertex outside the subcircuit of D rooted at v , then all certificates of D through v agree on setting $|M|$ variables to 1.

Corollary 1. *Let E be a graph CNF whose underlying graph is $G = (V, E)$ and let D be a nice DNNF equivalent to E . Let v be a gate in D and M be a matching in G between $\text{vars}(\text{sub}(D, v))$ and $\text{vars}(D) \setminus \text{vars}(\text{sub}(D, v))$. There exists $I_v \subseteq V$ such that $|I_v| = |M|$ and for all $T \in \text{cert}(D)$, if $v \in T$ then $I_v \subseteq \text{vars}(T)$.*

Proof. Let $e = \{x, x'\} \in M$ with $x \in \text{vars}(\text{sub}(D, v))$ and $x' \in \text{vars}(D) \setminus \text{vars}(\text{sub}(D, v))$. Since $x \vee x'$ is a clause in the CNF E , for all $T \in \text{cert}(D)$, either $x \in \text{vars}(T)$ or $x' \in \text{vars}(T)$. Thus by Theorem 2, either $x \in \text{vars}(T)$ for all $T \in \text{cert}(D)$ such that $v \in T$ or $x' \in \text{vars}(T)$ for all $T \in \text{cert}(D)$ such that $v \in T$. Let x_e be the vertex of e that is in every $T \in \text{cert}(D)$ such that $v \in T$. We choose $I_v = \{x_e \mid e \in M\}$.

By construction, it is clear that for all $T \in \text{cert}(D)$ such that $v \in T$, we have $I_v \subseteq \text{vars}(T)$. Moreover, since M is a matching, for $e, e' \in M$, if $e \neq e'$ then $e \cap e' = \emptyset$ and thus $x_e \neq x_{e'}$, that is $|I_v| = |M|$. \square

4.2 Bottleneck Argument

We are now ready to set up our bottleneck argument. In the sequel, D is a nice DNNF computing a graph CNF E whose underlying graph is an expander $G = (V, E)$. We define a subset B of gates of D (Lemma 2) and, for each gate $v \in B$, a subset I_v of V (Lemma 3) in such a way that the fraction of vertex covers containing I_v is exponentially small in $|V|$ (Corollary 2), hence B is exponentially large in $|V|$ (Theorem 4).

4.2.1 Finding the Bottleneck Gates.

We define the bottleneck $B \subseteq D$ as follows. For every certificate T of D we find (in a greedy fashion) a node v_T in T such that the subcircuit of D rooted at v_T has a large but not too large number of variables, and we put v_T into B .

Lemma 1. *Let E be a graph CNF whose underlying graph $G = (V, E)$ is connected and has degree d ($d \geq 3$). Let D be a nice DNNF equivalent to E and let $T \in \text{cert}(D)$. There exists a gate $v_T \in T$ such that*

$$|V|/(d+1) \leq |\text{vars}(\text{sub}(D, v_T))| \leq |V|/2. \quad (3)$$

Proof. We claim that $\frac{|V|}{(d+1)} \leq |\text{vars}(T)|$. Indeed, let f be the assignment defined by $f(v) = 1$ if and only if $v \in \text{vars}(T)$. Then f satisfies D by Proposition 1. Since D computes E , we have that $\text{vars}(T) = \{v : f(v) = 1\}$ is a vertex cover of G . Thus by Proposition 2, $|\text{vars}(T)| \geq |V|/(d+1)$.

By Proposition 3, with $\ell = |V|/(d+1)$, there exists a vertex v_T in T such that $|V|/(d+1) \leq |\text{vars}(\text{sub}(D, v_T))| \leq 2|V|/(d+1) \leq |V|/2$ where the last inequality comes from the fact that $d+1 \geq 4$. \square

Lemma 2. *Let E be a graph CNF whose underlying graph $G = (V, E)$ is connected and has degree d ($d \geq 3$). Let D be a nice DNNF equivalent to E . There exist $B \subseteq D$ such that:*

- (i) $|V|/(d+1) \leq |\text{vars}(\text{sub}(D, v))| \leq |V|/2$, for all $v \in B$.
- (ii) For all $T \in \text{cert}(D)$ there exists $v \in B$ such that $v \in T$.

Proof. We simply choose $B = \{v_T \mid T \in \text{cert}(D)\}$ where v_T is the vertex of T from Lemma 1. \square

4.2.2 Mapping the Vertex Covers.

For each $v \in B$, we find a large matching in G between variables inside and outside the subcircuit rooted at v , and then use Corollary 1 to derive a large set $I_v \subseteq V$ such that for all certificates T through v it holds that $I_v \subseteq \text{vars}(T)$.

Recall that a *matching* in a graph G is a subset M of the edges such that $\{u, v\} \cap \{u', v'\} \neq \emptyset$ for every two distinct edges $\{u, v\}$ and $\{u', v'\}$ in M . For disjoint subsets V' and V'' of the vertices of G , a matching M in G is said *between V' and V''* if every edge in M intersects both V' and V'' .

Lemma 3. *Let E be a graph CNF whose underlying graph $G = (V, E)$ is a (c, d) -expander ($d \geq 3$, $c > 0$), let D be a nice DNNF equivalent to E , and let $v \in D$ such that $|V|/(d+1) \leq |\text{vars}(\text{sub}(D, v))| \leq |V|/2$. There exists $I_v \subseteq V$ such that:*

- (i) For all $T \in \text{cert}(D)$ such that $v \in T$ it holds that $I_v \subseteq \text{vars}(T)$.
- (ii) $|I_v| \geq c|V|/(2d^2)$.

Proof. The idea is to construct a matching between $S = \text{vars}(\text{sub}(D, v))$ and $V \setminus S$ of size at least $c|V|/(2d^2)$ and then apply Corollary 1. Since $|V|/(d+1) \leq |S| \leq |V|/2$ and G is a (c, d) -expander, by (1) and (3) we have that

$$|N_S| \geq c|S| \geq c|V|/(d+1).$$

We construct a matching M between S and $V \setminus S$ in G as follows. Pick an edge $\{v, w\} \in E$ with $v \in S$ and $w \in N_S \subseteq V \setminus S$; add $\{v, w\}$ to M ; delete v from S , w from N_S , the vertices in S with no neighbors in N_S after the deletion of w , and the vertices in N_S with no neighbors in S after the deletion of v ; iterate on the updated S and N_S , until either $S = \emptyset$ or $N_S = \emptyset$. At each step, we delete at most d vertices in S and at most d vertices in N_S .

Hence, we iterate for at least

$$s \geq \min \left\{ \frac{|S|}{d}, \frac{|N_S|}{d} \right\} \geq \frac{\min\{1, c\}}{d(d+1)} |V| \geq \frac{c|V|}{2d^2}$$

steps ($d \geq 3$, $c \leq 1$). So we have that $|M| \geq s \geq c|V|/(2d^2)$, and we are done. Now, applying Corollary 1 on v and the matching M yields the result. \square

4.2.3 Proving the Lower Bound.

We conclude proving that for every $v \in B$ the fraction of vertex covers of G containing I_v is exponentially small (Corollary 2). On the other hand, by construction, every vertex cover of G contains a set I_v for some $v \in B$, so that the union (over $v \in B$) of the vertex covers of G containing I_v coincides with the vertex covers of G ; hence B is exponentially large (Theorem 4).

Let $G = (V, E)$ be a graph and let $S \subseteq V$. We denote by $\text{VC}(G, S)$ the set of vertex covers of G containing S .

Theorem 3. *Let $G = (V, E)$ be a graph, $S \subseteq V$, and $s \in S$. Then,*

$$|\text{VC}(G, S)| \leq \left(\frac{2^{d(s)}}{1 + 2^{d(s)}} \right) |\text{VC}(G, S \setminus \{s\})|.$$

Proof. Let $G = (V, E)$ be a graph, $S \subseteq V$, and $s \in S$. Let f be the mapping defined as $f(C) = (N_s \cap C, (C \setminus \{s\}) \cup N_s)$ for all $C \in \text{VC}(G, S)$. We denote by $\mathcal{P}(N_s)$ the power set of N_s , that is $\{A : A \subseteq N_s\}$.

First remark that for all $C \in \text{VC}(G, S)$, $f(C) = (A, D) \in \mathcal{P}(N_s) \times (\text{VC}(G, S \setminus \{s\}) \setminus \text{VC}(G, S))$. It is clear that $A = C \cap N_s \subseteq N_s$ thus $A \in \mathcal{P}(N_s)$. Moreover, if C is a vertex cover of G , then $D = (C \setminus \{s\}) \cup N_s$ is also a vertex cover of G , since each edge e of G is covered by C : If s is not an endpoint of e , then e is still covered by $C \setminus \{s\}$, and thus also by D . Otherwise $e = \{s, t\}$ with $t \in N_s$. Thus e is covered by D since $t \in D$. Finally, if $S \subseteq C$, then $S \setminus \{s\} \subseteq D$ and $s \notin D$. Thus $D \in \text{VC}(G, S \setminus \{s\})$ and $D \notin \text{VC}(G, S)$.

We now prove that f is an injection. Let $C, C' \in \text{VC}(G, S)$ such that $(A, D) = f(C) = f(C')$. Then, by definition, $C \cap N_s = C' \cap N_s$ and $C \setminus \{s\} \cup N_s = C' \setminus \{s\} \cup N_s$. Since s is both in C and in C' , we have $C \cup N_s = C' \cup N_s$ and $C \cap N_s = C' \cap N_s$, that is $C = C'$. Thus f is an injection. It follows that:

$$|\text{VC}(G, S)| \leq |\mathcal{P}(N_s)| \times |\text{VC}(G, S \setminus \{s\}) \setminus \text{VC}(G, S)|.$$

Since $\text{VC}(G, S) \subseteq \text{VC}(G, S \setminus \{s\})$, we have $|\text{VC}(G, S \setminus \{s\}) \setminus \text{VC}(G, S)| = |\text{VC}(G, S \setminus \{s\})| - |\text{VC}(G, S)|$ and it is clear that $|\mathcal{P}(N_s)| = 2^{d(s)}$. It follows that:

$$|\text{VC}(G, S)| \leq \left(\frac{2^{d(s)}}{1 + 2^{d(s)}} \right) |\text{VC}(G, S \setminus \{s\})|.$$

□

Corollary 2. *Let $G = (V, E)$ be a graph of degree d and let $S \subseteq V$. Then,*

$$|\text{VC}(G, S)| \leq \left(\frac{2^d}{1 + 2^d} \right)^{|S|} |\text{VC}(G)|.$$

Proof. Let $G = (V, E)$ be a graph of degree d , and let $S \subseteq V$. By induction on $|S| \geq 0$, we prove that

$$|\text{VC}(G, S)| \leq \left(\prod_{s \in S} \frac{2^{d(s)}}{1 + 2^{d(s)}} \right) |\text{VC}(G)|.$$

The statement follows since for all $s \in S$, we have $d(s) \leq d$ and thus:

$$\frac{2^{d(s)}}{1 + 2^{d(s)}} \leq \frac{2^d}{1 + 2^d}$$

The base case $S = \emptyset$ is trivial. If $S \neq \emptyset$, let $t \in S$. By Theorem 3,

$$|\text{VC}(G, S)| \leq \frac{2^{d(t)}}{1 + 2^{d(t)}} |\text{VC}(G, S \setminus \{t\})|$$

and thus by the induction hypothesis

$$|\text{VC}(G, S)| \leq \frac{2^{d(t)}}{1 + 2^{d(t)}} \left(\prod_{s \in S \setminus \{t\}} \frac{2^{d(s)}}{1 + 2^{d(s)}} \right) |\text{VC}(G)| = \left(\prod_{s \in S} \frac{2^{d(s)}}{1 + 2^{d(s)}} \right) |\text{VC}(G)|$$

□

Theorem 4. *Let $G = (V, E)$ be a (c, d) -expander such that $|V| \geq 2$. Then*

$$\text{DNNF}(E) \geq 2^{g(c, d) \text{size}(E) - 1},$$

where $g(c, d) = \frac{c \cdot f(d)}{6d^3}$ and $f(d) = \log_2(1 + 2^{-d}) > 0$.

Proof. Let D be a nice DNNF equivalent to E . By Proposition 4, we can assume that $\text{size}(D) \leq 2 \cdot \text{DNNF}(E)$.

Let $B \subseteq D$ be the set of gates from Lemma 2. Let $v \in B$. By construction, $|V|/(d+1) \leq |\text{vars}(\text{sub}(D, v))| \leq |V|/2$. Thus by Lemma 3, there exists I_v such that for all $T \in \text{cert}(D)$ such that $v \in T$, we have $I_v \subseteq \text{vars}(T)$ and $|I_v| \geq h(c, d)|V|$ where $h(c, d) = c/(2d^2)$. By Corollary 2,

$$|\text{VC}(G, I_v)| \leq \left(\frac{2^d}{1 + 2^d} \right)^{|I_v|} |\text{VC}(G)|.$$

As $f(d) = -\log_2\left(\frac{2^d}{1+2^d}\right)$, we have that, for all $v \in B$,

$$|\text{VC}(G, I_v)| \leq 2^{-f(d)h(c, d)|V|} |\text{VC}(G)|.$$

We claim that $\text{VC}(G) = \bigcup_{v \in B} \text{VC}(G, I_v)$. For the nontrivial containment, let $C \in \text{VC}(G)$ and, by Proposition 1, let $T \in \text{cert}(D)$ be such that $\text{vars}(T) \subseteq C$. By Lemma 2(ii), let $v \in B$ be such that $v \in T$. Then $I_v \subseteq \text{vars}(T)$ by Lemma 3(i), so that $I_v \subseteq C$, that is, $C \in \text{VC}(G, I_v)$. Therefore,

$$|\text{VC}(G)| \leq \sum_{v \in B} |\text{VC}(G, I_v)| \leq 2^{-f(d)h(c, d)|V|} |\text{VC}(G)| \cdot |B|,$$

from which $|B| \geq 2^{f(d)h(c, d)|V|}$.

Now observe that $|E| \leq d|V|$, because G has degree d . Thus, the CNF E has at most $d|V|$ clauses, each of at most 2 literals, so that $d|V| + 2d|V| = 3d|V| \geq \text{size}(E)$. Since $\text{size}(D) \geq |B|$ and $g(c, d) = \frac{f(d)h(c, d)}{3d}$, we finally have

$$\text{DNNF}(E) \geq \text{size}(D)/2 \geq 2^{g(c, d) \text{size}(E) - 1}.$$

□

Our main result follows from the previous theorem.

Theorem 5. *There exist a class \mathcal{C} of CNF formulas and a constant $c > 0$ such that $\text{DNNF}(F) \geq 2^{c \cdot \text{size}(F)}$ for each formula $F \in \mathcal{C}$. Indeed, \mathcal{C} is a class of read 3 times monotone 2-CNFs.*

Proof. By Theorem 1, there exists a family $\mathcal{G} = \{G_i = (V_i, E_i) : i \in \mathbb{N}\}$ of $(e, 3)$ -expander graphs such that $|V_i| \geq 2$ for all $i \in \mathbb{N}$ and $|V_i| \rightarrow \infty$ as $i \rightarrow \infty$ ($e > 0$). Every graph in \mathcal{G} is connected; in particular, it does not contain isolated vertices. Therefore E_i is a CNF for every $i \in \mathbb{N}$, and indeed E_i is a read 3 times monotone 2-CNF.

Since $|V_i| \rightarrow \infty$ as $i \rightarrow \infty$ and each graph in \mathcal{G} satisfies (1), there exists an infinite subset $I \subseteq \mathbb{N}$ such that $\text{size}(E_i) < \text{size}(E_{i+1})$ for all $i \in I$. Choose $c > 0$ and $j \in I$ large enough such that $g(e, 3) \cdot \text{size}(E_j) - 1 \geq c \cdot \text{size}(E_j)$, where $g(\cdot, \cdot)$ is as in the statement of Theorem 4. It follows from Theorem 4 that $\text{DNNF}(E_j) \geq 2^{g(e, 3) \cdot \text{size}(E_j) - 1} \geq 2^{c \cdot \text{size}(E_j)}$; we take $\mathcal{C} = \{E_i : i \in I \text{ and } i \geq j\}$, and the statement is proved. □

5 Corollaries

In this section we will prove the corollaries of Theorem 5 we sketched in the introduction.

Let F be a CNF. We say that a clause C is *entailed* by F if every satisfying assignment of F also satisfies C . We say that a clause C' *subsumes* C , if $C' \subseteq C$. A CNF F is in prime implicates form (short PI) if every clause that is entailed by F is subsumed by a clause that appears in F and no clause in F is subsumed by another. Note that CNFs in PI form can express all Boolean functions but it is known that encoding in PI form may generally be exponentially bigger than general CNF [DM02].

Lemma 4. *Every monotone 2-CNF formula is in PI form.*

Proof. Let F be a monotone 2-CNF formula. We first note that trivially no clause in a 2-CNF subsumes another.

Now let C be a clause entailed by F and assume by way of contradiction that C is not subsumed by any clause of F , that is, every clause in F contains a positive literal not in C . Let C' be the clause we get from C by deleting all negative literals. We claim that C' is entailed by F . To see this, consider a satisfying assignment f of F . Let f' be the assignment we get from f by setting the variables that are negated in C to 1. Since F is monotone, this is still a satisfying assignment of F and thus of C . Consequently, C is satisfied by one of its positive literals in f' and thus in f . Thus f satisfies C' and it follows that F entails C' .

Now let f be the assignment that sets all variables in C' to 0 and all other variables to 1. Since C and thus also C' is not subsumed by any clause of F , the assignment f satisfies F . But by construction f does not satisfy C' which is a contradiction. \square

Remember that the formulas of Theorem 5 are monotone 2-CNF formulas. We directly get the promised separation from Lemma 4 and Theorem 5.

Corollary 3. *There exist a class \mathcal{C} of CNFs in PI form and a constant $c > 0$ such that, for every formula F in \mathcal{C} , every DNNF equivalent to F has size at least $2^{c \cdot \text{size}(F)}$.*

It follows that L_1 can be exponentially more succinct than L_2 for any two representation languages $L_1 \supseteq \text{PI}$ and $L_2 \subseteq \text{DNNF}$. In particular, this holds if $L_1 \in \{\text{PI}, \text{CNF}, \text{NNF}\}$ and $L_2 \in \{\text{d-DNNF}, \text{DNNF}\}$. Here, d-DNNF denotes the language of *deterministic* DNNFs, that is, DNNFs where subcircuits leading into a \vee -gate never simultaneously evaluate to 1. This answers several questions concerning the relative succinctness of common representation languages [DM02].

We also observe that DNNFs are not closed under negation.

Lemma 5. *There exist a class \mathcal{D} of 2-DNF formulas and a constant $c > 0$ such that, for every formula D in \mathcal{D} , every DNNF equivalent to $\neg D$ has size at least $2^{c \cdot \text{size}(D)}$.*

Proof. Let \mathcal{C} be the class of 2-CNFs from Theorem 5. Let \mathcal{D} be the class of 2-DNFs we get by negating the formulas in \mathcal{C} . Now negating \mathcal{D} gives the class \mathcal{C} again, for which we have the lower bound from Theorem 5. \square

Observing that DNF is a restricted form of DNNF, we get the following non-closure result which was only known conditionally before.

Corollary 4. *There exist a class \mathcal{D} of DNNFs and a constant $c > 0$ such that, for every formula D in \mathcal{D} , every DNNF equivalent to $\neg D$ has size at least $2^{c \cdot \text{size}(D)}$.*

6 Conclusion

We proved an unconditional, strongly exponential separation between the representational power of CNFs and that of DNNFs and discussed its consequences in the area of knowledge compilation [DM02]. Let us close by mentioning directions for future research.

In order to prove the lower bound of Theorem 5, we generalized arguments concerning paths in branching programs to the tree-shaped certificates of DNNFs. It would be interesting to know whether other lower bounds for branching programs can be lifted to (suitably restricted) versions of DNNFs along similar lines.

Recent progress notwithstanding [BLRS13], several separations between well-known representation languages are known to hold only conditionally [DM02]. For instance, it is known that DNFs cannot be compiled efficiently into so-called deterministic DNNFs unless the polynomial hierarchy collapses [SK96, CDLS02]. It would be interesting to show this separation unconditionally.

Finally, there is a long line of research proving upper bounds for DNNFs and restrictions (see [Raz14a, OD14a, OD14b] for some recent contributions). We believe that these results should be complemented by lower bounds as in [Raz14a, Raz14b], and hope that the ideas developed in this paper will contribute to this project.

Acknowledgments. The first author was supported by the European Research Council (Complex Reason, 239962) and the FWF Austrian Science Fund (Parameterized Compilation, P26200). The third author has received partial support by a Qualcomm grant administered by École Polytechnique. The results of this paper were conceived during a research stay of the second and third author at the Vienna University of Technology. The stay of the second author was made possible by financial support by the ANR Blanc COMPA. The stay of the third author was made possible by financial support by the ANR Blanc International ALCOCLAN. The fourth author was supported by the European Research Council (Complex Reason, 239962).

References

- [AB87] N. Alon and R. B. Boppana. The monotone circuit complexity of boolean functions. *Combinatorica*, 7(1):1–22, 1987.
- [AS00] N. Alon and J. H. Spencer. *The Probabilistic Method*. Wiley, 2000.
- [BLRS13] P. Beame, J. Li, S. Roy, and D. Suciu. Lower Bounds for Exact Model Counting and Applications in Probabilistic Databases. In *Proceedings of CUA1*, 2013.
- [BLRS14] P. Beame, J. Li, S. Roy, and D. Suciu. Counting of Query Expressions: Limitations of Propositional Methods. In *Proceeding of ICDT*, pages 177–188, 2014.
- [BS14] S. Bova and F. Slivovsky. On Compiling Structured CNFs to OBDDs. *CoRR*, abs/1411.5494, 2014.
- [CDLS02] M. Cadoli, F. Donini, P. Liberatore, and M. Schaerf. Preprocessing of Intractable Problems. *Information and Computation*, 176:89–120, 2002.
- [CM78] A. Chandra and G. Markowsky. On the Number of Prime Implicants. *Discrete Mathematics*, 24:7–11, 1978.
- [Dar01] A. Darwiche. Decomposable negation normal form. *J. ACM*, 48(4):608–647, 2001.
- [Die05] Reinhard Diestel. *Graph Theory*. Springer, August 2005.
- [DM02] A. Darwiche and P. Marquis. A Knowledge Compilation Map. *J. Artif. Intell. Res. (JAIR)*, 17:229–264, 2002.
- [GKPS95] G. Gogic, H. A. Kautz, C. H. Papadimitriou, and B. Selman. The Comparative Linguistics of Knowledge Representation. In *Proceedings of IJCAI*, 1995.
- [Hak85] A. Haken. The intractability of resolution. *Theor. Comput. Sci.*, 39:297–308, 1985.

- [Juk12] Stasys Jukna. *Boolean Function Complexity - Advances and Frontiers*, volume 27 of *Algorithms and combinatorics*. Springer, 2012.
- [Kri07] Matthias P. Krieger. On the incompressibility of monotone dnfs. *Theory Comput. Syst.*, 41(2):211–231, 2007.
- [OD14a] U. Oztok and A. Darwiche. CV-width: A New Complexity Parameter for CNFs. In *Proceedings of ECAI*, pages 675–680, 2014.
- [OD14b] U. Oztok and A. Darwiche. On Compiling CNF into Decision-DNNF. In *Proceedings of CP*, 2014.
- [PD10] T. Pipatsrisawat and A. Darwiche. A Lower Bound on the Size of Decomposable Negation Normal Form. In *Proceedings of AAAI*, 2010.
- [PV04] A.K. Ponnuswami and H. Venkateswaran. Monotone Multilinear Boolean Circuits for Bipartite Perfect Matching Require Exponential Size. In *Foundations of Software Technology and Theoretical Computer Science*, volume 3328. Springer, 2004.
- [Raz14a] Igor Razgon. On OBDDs for CNFs of Bounded Treewidth. In *Proceedings of KR*, 2014.
- [Raz14b] Igor Razgon. On the read-once property of branching programs and CNFs of bounded treewidth. *CoRR*, abs/1411.0264, 2014.
- [SK96] B. Selman and H. Kautz. Knowledge Compilation and Theory Approximation. *Journal of the ACM*, 43:193–224, 1996.
- [SV94] R. Sengupta and H. Venkateswaran. Multilinearity Can Be Exponentially Restrictive. *manuscript*, 1994.
- [Weg00] Ingo Wegener. *Branching Programs and Binary Decision Diagrams*. SIAM, 2000.